

Entanglement Entropy of Quantum Wire Junctions

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Abstract

We consider a fermion gas on a star graph modeling a quantum wire junction and derive the entanglement entropy of one edge with respect to the rest of the junction. The gas is free in the bulk of the graph, the interaction being localized in its vertex and described by a non-trivial scattering matrix. We discuss all point-like interactions, which lead to unitary time evolution of the system. We show that for a finite number of particles N , the Rényi entanglement entropies of one edge grow as $\ln N$ with a calculable prefactor, which depends not only on the central charge, but also on the total transmission probability from the considered edge to the rest of the graph. This result is extended to the case with an harmonic potential in the bulk.

1 Introduction

Quantum field theory on graphs attracted recently much attention mainly in relation with the study [1]-[6] of the transport properties of quantum wire networks. Different frameworks [7]-[20] have been developed to investigate the phase diagram and the conductance of these structures. Despite of the fact that the universal properties in the bulk are described by the well known Luttinger liquid theory, the different boundary conditions at the junctions lead to exotic phase diagrams [10, 11, 12, 15, 16, 17, 18, 19] whose degree of universality is not completely understood and is still under investigation. The results, concerning the charge transport, confirm that the conductance properties of the quantum wire networks are strongly affected by the boundary conditions as well.

In the present paper we analyze another physical quantity - the entanglement entropy of one edge of the junction with respect to all the others edges. Lots of studies on the entanglement properties of many-body systems in the last decade have unveiled new (universal) features of these systems and somehow put their global understanding on a deeper level (see e.g. the reviews [21]). In particular, von Neumann and Rényi entanglement entropies of the reduced density matrix ρ_A of a subsystem A turned out to be particularly useful for 1D systems. Rényi entanglement entropies are defined as

$$S^{(\alpha)} = \frac{1}{1-\alpha} \ln \text{Tr} \rho_A^\alpha. \quad (1.1)$$

For $\alpha \rightarrow 1$ this definition gives the most commonly used von Neumann entropy $S^{(1)} = -\text{Tr} \rho_A \ln \rho_A$, while for $\alpha \rightarrow \infty$ is the logarithm of the largest eigenvalue of ρ_A also known as single copy entanglement [22]. Furthermore, the knowledge of the $S^{(\alpha)}$ for different α characterizes the full spectrum of non-zero eigenvalues of ρ_A [23].

One of the most remarkable results is the universal behavior displayed by the entanglement entropy at 1D conformal quantum critical points, determined by the central charge [24] of the underlying conformal field theory (CFT) [25, 26, 27, 28]. For a partition of an infinite 1D system into a finite piece A of length ℓ and the remainder, the Rényi entanglement entropies for ℓ much larger than the short-distance cutoff a are

$$S^{(\alpha)} = \frac{c}{6} \left(1 + \frac{1}{\alpha} \right) \ln \frac{\ell}{a} + c_\alpha, \quad (1.2)$$

where c is the central charge and c_α a non-universal constant.

Given the importance of this result (and also many others not mentioned here) for homogeneous systems, it is natural to wonder whether in the case of junctions the entanglement entropies can share some light on the universality and on the relevance of the parameters defining the junction. Previous studies in the subject [29, 30, 31, 32, 33, 34, 35, 36, 37] have been limited to the case of only two edges (i.e. an infinite line with a defect) and most often performed for lattice models. These results provide a strong evidence that the logarithmic behavior in Eq. (1.2) remains valid even in the presence of defects, but the prefactor does not depend

only on the central charge of the bulk CFT when the defect is a *marginal* perturbation (in renormalization group sense) as it is known to happen for free fermions [1].

In order to tackle the problem of entanglement in a junction with an arbitrary number of wires, we use the recently developed systematic framework [38, 39] for calculating the bipartite entanglement entropy of spatial subsystems of one-dimensional quantum systems in continuous space. We only consider a free fermion gas in bulk in which the junction introduces a marginal perturbation. The junction boundary conditions define a specific scattering matrix \mathbb{S} , encoding all possible point-like interactions in the vertex which give rise to unitary time evolution. Focussing on the scale invariant case, we show that for a finite number of particles N and for edges of equal length L , the Rényi entanglement entropies of any of the edges grow as $\ln N$. Oppositely to the case in the absence of the point-like interaction (i.e. Eq. (1.2)), the prefactor of this logarithm does not depend only on the central charge, but also on the total transmission probability $(1 - |\mathbb{S}_{ii}|^2)$ from the considered edge i to the rest of the graph. Some of the results presented here have been anticipated in the short communication [38]. We show also that the presence of an external harmonic potential in the bulk (acting identically on all edges) does not alter this result.

The paper is organized as follows. In the next section we describe the basic features of the model and the scattering matrices generated by the point-like interactions at the junction. We discuss in detail the scale invariant case and derive the two-point correlation function. The entanglement entropy, associated with this system, is analytically computed in section 3. In section 4 we extend our considerations, adding a harmonic potential in the bulk. Section 5 is devoted to the conclusions and the discussion of some further developments in the subject.

2 Schrödinger junction

2.1 The general setting

In this section we consider a gas of N spinless fermions on a quantum wire junction. We consider only the ground-state of such system and we refer to it as the ground-state of N *particles*, having in mind that the particles are the original fermions and not the excitations above the ground-state (that are usually referred as particles in field theory literature). A simple model, describing the junction, is represented by a star graph Γ with M edges of finite length L , as shown in Fig. 1. Each point P in the bulk of Γ is parametrized by (x, i) , where $0 \leq x \leq L$ is the distance of P from the vertex V of the graph and i labels the edge. We assume that in the bulk ($x \neq 0$ and $x \neq L$) the gas is free and is described by the Schrödinger field $\psi_i(t, x)$, which satisfies

$$\left(i\partial_t + \frac{1}{2m}\partial_x^2\right)\psi_i(t, x) = 0 \quad (2.1)$$

and standard equal-time canonical anticommutation relations. The only non-trivial interactions are localized in the vertex V of Γ and are encoded in boundary conditions at $x = 0$. These conditions are fixed in turn by imposing that the bulk Hamiltonian $-\partial_x^2$ admits a self-adjoint extension on the whole graph. In such a way all point-like interactions, leading to a unitary time

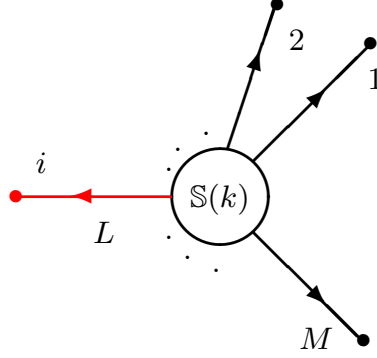


Figure 1: (Color online) A star graph Γ with scattering matrix $\mathbb{S}(k)$ in the vertex and all edges of length L . We consider the entanglement entropy of the edge i (red) with respect to all the others.

evolution of the system, are covered. The most general boundary conditions, implementing this natural physical requirement, are [41, 42] at the vertex

$$\sum_{j=1}^M [\lambda(\mathbb{I} - \mathbb{U})_{ij} \psi_j(t, 0) - i(\mathbb{I} + \mathbb{U})_{ij} (\partial_x \psi_j)(t, 0)] = 0, \quad (2.2)$$

where \mathbb{U} is an arbitrary $M \times M$ unitary matrix and λ a real parameter with the dimension of mass. To fully specify the problem we also need to impose boundary conditions at the external ends of the edges. The most general ones are

$$(\partial_x \psi_i)(t, L) = \mu_i \psi_i(t, L), \quad (2.3)$$

where μ_i are again real parameters with the dimension of mass. Eq. (2.3) is the familiar Robin (mixed) boundary condition. Notice that Eq. (2.2) extends this condition to the vertex V of the graph.

It has been established in [41, 42] that the point-like interaction, induced by (2.2), generates the scattering matrix

$$\mathbb{S}(k) = -\frac{[\lambda(\mathbb{I} - \mathbb{U}) - k(\mathbb{I} + \mathbb{U})]}{[\lambda(\mathbb{I} - \mathbb{U}) + k(\mathbb{I} + \mathbb{U})]}. \quad (2.4)$$

Besides of unitarity

$$\mathbb{S}(k)\mathbb{S}^\dagger(k) = \mathbb{I}, \quad (2.5)$$

$\mathbb{S}(k)$ satisfies Hermitian analyticity

$$\mathbb{S}^\dagger(k) = \mathbb{S}(-k) \quad (2.6)$$

as well. Notice also that

$$\mathbb{S}(\lambda) = \mathbb{U}, \quad \mathbb{S}(-\lambda) = \mathbb{U}^{-1}, \quad (2.7)$$

showing that the unitary matrix \mathbb{U} , entering the boundary conditions (2.2), is actually the scattering matrix at the scale λ .

The main difficulty in solving the Schrödinger equation (2.1) on the graph Γ is the mixing between the different edges codified in the boundary conditions (2.2) and (2.3). In order to simplify this problem, we impose that the boundary conditions at the ends of each arm are all the same, in such a way to restore (at the level of the Hamiltonian) permutation symmetry of the edges of the graph. It should be clear from the physical point of view that, being interested in the thermodynamic limit with $L, N \rightarrow \infty$, the boundary conditions at L must not affect the final result. Thus we assume from now on that

$$\mu_1 = \mu_2 = \dots = \mu_M \equiv \mu. \quad (2.8)$$

Under this condition, Eqs. (2.2) and (2.3) can be rewritten in equivalent forms without mixing. Indeed, let us introduce the unitary matrix \mathcal{U} diagonalizing \mathbb{U} , namely

$$\mathcal{U} \mathbb{U} \mathcal{U}^\dagger = \mathbb{U}_d = \text{diag} \left(e^{-2i\alpha_1}, e^{-2i\alpha_2}, \dots, e^{-2i\alpha_M} \right), \quad -\frac{\pi}{2} < \alpha_i \leq \frac{\pi}{2}. \quad (2.9)$$

Remarkably enough, \mathcal{U} diagonalizes also $\mathbb{S}(k)$ for *any* k :

$$\mathbb{S}_d(k) = \mathcal{U}^\dagger \mathbb{S}(k) \mathcal{U} = \text{diag} \left(\frac{k + i\eta_1}{k - i\eta_1}, \frac{k + i\eta_2}{k - i\eta_2}, \dots, \frac{k + i\eta_M}{k - i\eta_M} \right), \quad (2.10)$$

where

$$\eta_i \equiv \lambda \tan(\alpha_i). \quad (2.11)$$

It is quite natural at this point to introduce the fields

$$\varphi_i(t, x) = \sum_{j=1}^M \mathcal{U}_{ij} \psi_j(t, x), \quad (2.12)$$

which obviously satisfy Eq. (2.1). In terms of φ_i the boundary conditions (2.2, 2.3) decouple,

$$(\partial_x \varphi_i)(t, 0) = \eta_i \varphi_i(t, 0), \quad (2.13)$$

$$(\partial_x \varphi_i)(t, L) = \mu \varphi_i(t, L), \quad (2.14)$$

defining a simple spectral problem on the tensor product $\mathcal{H} = \bigotimes_{i=1}^M \mathbb{L}^2[0, L]$, which is analyzed below.

It is worth stressing that $\varphi_i(t, x)$ is a superposition of the values of the original field $\psi_i(t, x)$ at the same distance x from the vertex, but on *different* edges of the junction. Being so delocalized, $\varphi_i(t, x)$ is unphysical and provides only a convenient basis for dealing with the boundary conditions. The physical observables and correlation functions will be always expressed in terms of the physical fields $\psi_i(t, x)$.

The eigenfunctions of $-\partial_x^2$, obeying (2.13) and (2.14) are

$$\phi_i(k, x) = c_i \left(e^{ikx} + \frac{k + i\eta_i}{k - i\eta_i} e^{-ikx} \right), \quad k \geq 0, \quad (2.15)$$

where c_i are some constants to be fixed below and k satisfy

$$e^{2ikL} = \left(\frac{k + i\eta_i}{k - i\eta_i} \right) \left(\frac{k - i\mu}{k + i\mu} \right). \quad (2.16)$$

In order to determine the spectrum of k explicitly, we simplify the problem further by requiring scale invariance.

2.2 The scale invariant case

Scale invariance of the boundary conditions (2.13) and (2.14) implies the values

$$\mu = \begin{cases} 0, \\ \infty, \end{cases} \quad \eta_i = \begin{cases} 0 & (\alpha_i = 0), \\ \infty & (\alpha_i = \pi/2), \end{cases} \quad \begin{array}{l} \text{Neumann b.c.}, \\ \text{Dirichlet b.c.} \end{array} \quad (2.17)$$

In other words, the critical points are fixed by the $(M + 1)$ vector $(\mu, \eta_1, \eta_2, \dots, \eta_M)$ whose components take the values (2.17). For an edge i one has the following possibilities:

(a) $\mu = 0$ (Neumann condition at $x = L$): Eq. (2.16) gives

$$e^{2ikL} = \left(\frac{k + i\eta_i}{k - i\eta_i} \right), \quad (2.18)$$

which gives

$$\eta_i = 0 \implies \phi_i(n, x) = \sqrt{\frac{2}{L}} \cos \left[(n - 1) \pi \frac{x}{L} \right], \quad n = 1, 2, \dots \quad (2.19)$$

$$\eta_i = \infty \implies \phi_i(n, x) = \sqrt{\frac{2}{L}} \sin \left[\left(n - \frac{1}{2} \right) \pi \frac{x}{L} \right], \quad n = 1, 2, \dots; \quad (2.20)$$

(b) $\mu = \infty$ (Dirichlet condition at $x = L$): Eq. (2.16) implies

$$e^{2ikL} = - \left(\frac{k + i\eta_i}{k - i\eta_i} \right), \quad (2.21)$$

which gives

$$\eta_i = 0 \implies \phi_i(n, x) = \sqrt{\frac{2}{L}} \cos \left[\left(n - \frac{1}{2} \right) \pi \frac{x}{L} \right], \quad n = 1, 2, \dots \quad (2.22)$$

$$\eta_i = \infty \implies \phi_i(n, x) = \sqrt{\frac{2}{L}} \sin \left(n \pi \frac{x}{L} \right), \quad n = 1, 2, \dots \quad (2.23)$$

Notice that any of the sets (2.18, 2.19, 2.22, 2.23) represent a complete ortho-normal system in $\mathbb{L}^2[0, L]$.

2.3 Scale invariant scattering matrices

Observing that the eigenvalue of \mathbb{S} is 1 for $\eta_i = 0$ and -1 for $\eta_i = \infty$ one concludes that the most general scale-invariant scattering matrix, compatible with a unitary time evolution, is given by

$$\mathbb{S} = \mathcal{U} \mathbb{S}_d \mathcal{U}^\dagger, \quad \mathbb{S}_d = \text{diag}(\pm 1, \pm 1, \dots, \pm 1), \quad (2.24)$$

where \mathcal{U} is a generic $M \times M$ unitary matrix. From the group-theoretical point of view, any critical \mathbb{S} matrix is a point in the orbit of some \mathbb{S}_d under the *adjoint* action of the unitary group $U(M)$. Obviously, one can enumerate the edges in such a way that the first p eigenvalues of \mathbb{S} are $+1$ and the remaining $M - p$ are -1 . The cases $p = M$ and $p = 0$ correspond to $\mathbb{S} = \mathbb{I}$ and $\mathbb{S} = -\mathbb{I}$ and are not interesting for the entanglement. In these two cases in fact, the single wires are decoupled (there is no transmission), which implies a vanishing entanglement.

It follows from (2.24) that besides being unitary, \mathbb{S} is also Hermitian (in agreement with (2.6) at criticality). Therefore, all diagonal elements \mathbb{S}_{ii} are real. Notice however that in general \mathbb{S} is not symmetric. If this is the case, time reversal invariance is broken [19].

In the nontrivial case $p = 1$ the most general 2×2 scale-invariant scattering matrix depends on two parameters and can be written in the form

$$\mathbb{S}(\epsilon, \theta) = \frac{1}{1 + \epsilon^2} \begin{pmatrix} \epsilon^2 - 1 & 2\epsilon e^{i\theta} \\ 2\epsilon e^{-i\theta} & 1 - \epsilon^2 \end{pmatrix}, \quad \epsilon \in \mathbb{R}, \quad \theta \in [0, 2\pi). \quad (2.25)$$

Time reversal invariance is broken for $\theta \neq 0, \pi$.

For $M = 3$ one has two families corresponding to $p = 1$ and $p = 2$. In order to avoid cumbersome formulae, we display only two representatives of these families, namely

$$\mathbb{S}_{p=1}(\epsilon_1, \epsilon_2) = \frac{1}{1 + \epsilon_1^2 + \epsilon_2^2} \begin{pmatrix} 2\epsilon_1 & 2\epsilon_2 & 1 - \epsilon_1^2 - \epsilon_2^2 \\ 2\epsilon_2 & -\epsilon_1^2 + \epsilon_2^2 - 1 & 2\epsilon_1\epsilon_2 \\ \epsilon_1^2 - \epsilon_2^2 - 1 & 2\epsilon_1\epsilon_2 & 2\epsilon_1 \end{pmatrix}, \quad (2.26)$$

$$\mathbb{S}_{p=2}(\epsilon_1, \epsilon_2) = \frac{-1}{1 + \epsilon_1^2 + \epsilon_2^2} \begin{pmatrix} \epsilon_1^2 - \epsilon_2^2 - 1 & 2\epsilon_1\epsilon_2 & 2\epsilon_1 \\ 2\epsilon_1\epsilon_2 & -\epsilon_1^2 + \epsilon_2^2 - 1 & 2\epsilon_2 \\ 2\epsilon_1 & 2\epsilon_2 & 1 - \epsilon_1^2 - \epsilon_2^2 \end{pmatrix}, \quad (2.27)$$

where $\epsilon_{1,2} \in \mathbb{R}$.

2.4 Two-point correlation function

Now we are in position to construct the physical field $\psi_i(t, x)$ and the relative two-point function needed in the computation of the entanglement entropy. First of all, we write the unphysical field in terms of the eigenfunctions $\phi_i(n, x)$

$$\varphi_i(t, x) = \sum_{n=1}^{\infty} e^{-i\omega_i(n)t} \phi_i(n, x) a_i(n), \quad (2.28)$$

where the fermion annihilation and creation operators satisfy standard anti-commutation relations

$$[a_i(m), a_j^\dagger(n)]_+ = \delta_{ij}\delta_{mn}, \quad [a_i(m), a_j(n)]_+ = [a_i^\dagger(m), a_j^\dagger(n)]_+ = 0, \quad (2.29)$$

and the energies are given by

$$\omega_i(n) = \begin{cases} \frac{1}{2m} \left[(n-1)\frac{\pi}{L} \right]^2, & \text{if } 1 \leq i \leq p, \\ \frac{1}{2m} \left[(2n-1)\frac{\pi}{2L} \right]^2, & \text{if } p < i \leq M, \end{cases} \quad (2.30)$$

$$\omega_i(n) = \begin{cases} \frac{1}{2m} \left[(2n-1)\frac{\pi}{2L} \right]^2, & \text{if } 1 \leq i \leq p, \\ \frac{1}{2m} \left[n\frac{\pi}{L} \right]^2, & \text{if } p < i \leq M, \end{cases} \quad (2.31)$$

for $\mu = 0$ and $\mu = \infty$ respectively. Notice that different “unphysical” edges may have different dispersion relation, which is not a problem because these edges are totally isolated from each other. By means of (2.12) one gets the physical fields

$$\psi_i(t, x) = \sum_{j=1}^M \mathcal{U}_{ij}^\dagger \varphi_j(t, x) = \sum_{j=1}^M \sum_{n=1}^{\infty} \mathcal{U}_{ij}^\dagger e^{-i\omega_j(n)t} \phi_j(n, x) a_j(n). \quad (2.32)$$

One easily verifies that

$$[\psi_i(t, x), \psi_j^\dagger(t, y)]_+ = \delta_{ij}\delta(x-y), \quad (2.33)$$

which fixes the normalization of the fields.

The equal time two-point correlation function of the physical field $\psi_i(t, x)$ on a given state $|\Psi\rangle$ is

$$\begin{aligned} C_{ij}^\Psi(x, y) &\equiv \langle \Psi | \psi_i^\dagger(t, x) \psi_j(t, y) | \Psi \rangle \\ &= \sum_{k,l=1}^M \sum_{n,m=1}^{\infty} \mathcal{U}_{jk}^\dagger \mathcal{U}_{li} e^{i[\omega_j(m) - \omega_i(n)]t} \phi_k(n, x) \bar{\phi}_l(m, y) \langle \Psi | a_k^\dagger(n) a_l(m) | \Psi \rangle, \end{aligned} \quad (2.34)$$

where the correlator $\langle \Psi | a_k^\dagger(n) a_l(m) | \Psi \rangle$ can be deduced from the action of the algebra generated by $\{a_i(m), a_j^\dagger(n)\}$ on the state $|\Psi\rangle$. In particular, we are interested in the case when $|\Psi\rangle$ is the ground-state of the system formed by N fermions in the whole junction. It is then useful to rewrite N as

$$N = M\mathcal{N} \quad (2.35)$$

where \mathcal{N} represents the average number of particles for each wire. The action of the annihilation and creation operators on the ground-state is obvious since it is annihilated by all $a_l(m)$ with $m > \mathcal{N}$ and so $\langle \Psi | a_k^\dagger(n) a_l(m) | \Psi \rangle = \delta_{kl}\delta_{nm}\theta(\mathcal{N} - n)$. Using this relation, Eq. (2.34), restricted to the same edge ($i = j$) which is needed actually for computing the entanglement entropy, becomes

$$C_{ii}^N(x, y) = \sum_{k=1}^M \sum_{n=1}^{\mathcal{N}} |\mathcal{U}_{ki}|^2 \phi_k(n, x) \bar{\phi}_k(n, y), \quad (2.36)$$

where \mathcal{N} can also be interpreted as an ultraviolet cut-off for the series in (2.34).

It is convenient for what follows to rewrite (2.36) in more explicit terms. For this purpose we consider any critical point characterized by the integer $1 < p < M$, i.e. a scale invariant scattering matrix with p eigenvalues equal to +1. The two sums in (2.36) factorize and one gets

$$C_{ii}^N(x, y) = \sum_{k=1}^p |\mathcal{U}_{ki}|^2 \sum_{n=1}^{\mathcal{N}} f_+(n, x) f_+(n, y) + \sum_{k=p+1}^M |\mathcal{U}_{ki}|^2 \sum_{n=1}^{\mathcal{N}} f_-(n, x) f_-(n, y), \quad (2.37)$$

where

$$f_+(n, x) = \begin{cases} \sqrt{\frac{2}{L}} \cos \left[(n-1) \pi \frac{x}{L} \right], & \mu = 0, \\ \sqrt{\frac{2}{L}} \cos \left[\left(n - \frac{1}{2}\right) \pi \frac{x}{L} \right], & \mu = \infty, \end{cases} \quad (2.38)$$

$$f_-(n, x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \left[\left(n - \frac{1}{2}\right) \pi \frac{x}{L} \right], & \mu = 0, \\ \sqrt{\frac{2}{L}} \sin \left[n \pi \frac{x}{L} \right], & \mu = \infty. \end{cases} \quad (2.39)$$

Because of (2.24), the sums over k in (2.37) give

$$\sum_{k=1}^p |\mathcal{U}_{ki}|^2 = \frac{1}{2} (1 + \mathbb{S}_{ii}), \quad \sum_{k=p+1}^M |\mathcal{U}_{ki}|^2 = \frac{1}{2} (1 - \mathbb{S}_{ii}). \quad (2.40)$$

Moreover, using that \mathbb{S} is both unitary and Hermitian, one has

$$\mathbb{S}_{ii}^2 = 1 - \sum_{\substack{j=1 \\ j \neq i}}^M |\mathbb{S}_{ij}|^2 \equiv 1 - T_i^2, \quad (2.41)$$

where T_i^2 is the *total transmission probability* from the edge i to the rest of the graph.

In conclusion, the correlator (2.36) can be fully expressed in terms of the transmission probability T_i^2 and the one-particle wave functions as follows

$$C_{ii}^N(x, y) = \frac{1}{2} \left(1 + \sqrt{1 - T_i^2} \right) \sum_{n=1}^{\mathcal{N}} f_+(x, n) f_+(y, n) + \frac{1}{2} \left(1 - \sqrt{1 - T_i^2} \right) \sum_{n=1}^{\mathcal{N}} f_-(x, n) f_-(y, n), \quad (2.42)$$

which is the basic input for deriving the entanglement entropy in the next section. We observe that C_{ii}^N involves only the diagonal elements of \mathbb{S} and consequently, does not depend on the behavior of \mathbb{S} under transposition. Therefore, contrary to the conductance [19], the entanglement entropy in our case is not sensitive to the breaking of time-reversal invariance.

3 Entanglement entropy

In order to compute the bipartite Rényi entanglement entropies defined as in Eq. (1.1) of a subsystem A in the ground-state our star graph, we use the method recently introduced in

Refs. [38, 39]. The starting point to deal with a system made of a finite number of particles in continuous space is the Fredholm determinant

$$\mathbb{D}_A(\lambda) = \det [\lambda \delta_A(x, y) - C_A(x, y)] , \quad (3.43)$$

where $C_A(x, y)$ is the restriction of the correlation matrix $C(x, y)$ defined in Eq. (2.34) to A , i.e. $C_A = P_A C P_A$, where P_A is the projector on A . The same definition holds for $\delta_A(x, y) = P_A \delta(x - y) P_A$. Following the ideas for the lattice model [43], $\mathbb{D}_A(\lambda)$ can be introduced in such a way that it is a polynomial in λ having as zeros the eigenvalues of C_A . Since we are dealing only with free fermions in the bulk, the reduced density matrix ρ_A is Gaussian [44] and so one can easily derive [43, 38, 39]

$$S^{(\alpha)} \equiv \frac{\ln \text{Tr} \rho_A^\alpha}{1 - \alpha} = \oint \frac{d\lambda}{2\pi i} e_\alpha(\lambda) \frac{d \ln \mathbb{D}_A(\lambda)}{d\lambda}, \quad (3.44)$$

where the integration contour encircles the segment $[0, 1]$, and

$$e_\alpha(\lambda) = \frac{1}{1 - \alpha} \ln [\lambda^\alpha + (1 - \lambda)^\alpha] . \quad (3.45)$$

For $\alpha \rightarrow 1$, $e_1(\lambda) = -x \ln x - (1 - x) \ln(1 - x)$ and Eq. (3.44) gives the von Neumann entropy.

The Fredholm determinant is turned into a standard one by introducing the *reduced overlap* matrix \mathbb{A} (also considered in Ref. [45]) with elements

$$\mathbb{A}_{nm} = \int_{x_1}^{x_2} dz \bar{\phi}_n(z) \phi_m(z), \quad n, m = 1, \dots, D, \quad (3.46)$$

where in general $\phi_n(x)$ represent the eigenfuctions corresponding to the D lowest energy level which are occupied in the ground-state of the system with D degrees of freedom. The matrix \mathbb{A} satisfies $\text{Tr} C_A^k = \text{Tr} \mathbb{A}^k$ and so [39]

$$\ln \mathbb{D}_A(\lambda) = - \sum_{k=1}^{\infty} \frac{\text{Tr} C_A^k}{k \lambda^k} = - \sum_{k=1}^{\infty} \frac{\text{Tr} \mathbb{A}^k}{k \lambda^k} = \ln \det [\lambda \mathbb{I} - \mathbb{A}] = \sum_{m=1}^D \ln(\lambda - a_m), \quad (3.47)$$

where a_m are the eigenvalues of \mathbb{A} and D is its dimension to be specified later. Inserting (3.47) in the integral (3.44), we obtain

$$S^{(\alpha)} = \oint \frac{d\lambda}{2\pi i} \sum_{m=1}^D \frac{e_\alpha(\lambda)}{\lambda - a_m} = \sum_{m=1}^D e_\alpha(a_m) = \frac{1}{1 - \alpha} \text{Tr} \ln [\mathbb{A}^\alpha + (\mathbb{I} - \mathbb{A})^\alpha] , \quad (3.48)$$

as a consequence of the residue theorem.

In the following we will be interested only in the entanglement entropy of any edge i of the wire with respect to the rest of the junction in the global ground-state of the star graph. As we have seen above in Eq. (2.42), the two-point correlation function for finite number of particles N in the full star graph, can be written in the form (we omit the edge index i hereafter)

$$C^N(x, y) = \sum_{n=1}^{2\mathcal{N}} \bar{\chi}(x, n) \chi(y, n), \quad (3.49)$$

where $\chi(x, n)$ are proportional to the one-particle eigenfunctions $f_{\pm}(x, n)$ in Eq. (2.42). Then the Rényi entanglement entropy of the subsystem represented by a single wire is given by Eq. (3.48) where the eigenvalues a_m of \mathbb{A} are numerically calculated from the overlap matrix built with the correlation function above.

3.1 The case $M = 2$

It is instructive to consider first the case with two edges only. In this case, one has actually a segment of length $2L$ with a point-like conformal defect placed in the middle. This situation has been investigated on the lattice [35, 36] by other methods and represents a useful check for our framework. We set $L = 1$ for simplicity and consider the case $\mu = \infty$ (Dirichlet condition at $x = L$).¹ Combining the explicit form (2.25) of the \mathbb{S} -matrix with (2.41), one obtains the following transmission amplitudess

$$T_1 = T_2 = \frac{2\epsilon}{1 + \epsilon^2} \equiv T, \quad (3.50)$$

which are the square root of the transmission probability. Accordingly,

$$C_{11}^N(x, y) = C_{22}^N(x, y) = \sum_{n=1}^{2\mathcal{N}} \bar{\chi}(x, n) \chi(y, n), \quad (3.51)$$

with

$$\chi(k, x) = \begin{cases} \frac{\epsilon}{\sqrt{1+\epsilon^2}} \sqrt{2} \cos\left(k\pi \frac{x}{2}\right), & k = 1, 3, \dots, 2\mathcal{N} - 1, \\ \frac{1}{\sqrt{1+\epsilon^2}} \sqrt{2} \sin\left(k\pi \frac{x}{2}\right), & k = 2, 4, \dots, 2\mathcal{N}, \end{cases} \quad (3.52)$$

where $N = 2\mathcal{N}$ is the total particle number according to eq. (2.35). The matrix \mathbb{A} , defined by (3.46), reads in our case

$$\mathbb{A}_{mn} = \frac{\epsilon^2}{(1 + \epsilon^2)} \delta_{mn}, \quad m, n - \text{odd}, \quad (3.53)$$

$$\mathbb{A}_{mn} = \frac{1}{(1 + \epsilon^2)} \delta_{mn}, \quad m, n - \text{even}, \quad (3.54)$$

$$\mathbb{A}_{mn} = \frac{2\epsilon}{(1 + \epsilon^2)} \frac{2n}{\pi(n^2 - m^2)}, \quad m - \text{odd}, n - \text{even}, \quad (3.55)$$

$$\mathbb{A}_{mn} = \frac{2\epsilon}{(1 + \epsilon^2)} \frac{2m}{\pi(m^2 - n^2)}, \quad m - \text{even}, n - \text{odd}. \quad (3.56)$$

Since $C_{11}^N = C_{22}^N$, the entanglement entropy of the edge 1 equals that of the edge 2 and is given by

$$S^{(\alpha)}(T; N) = \frac{1}{1 - \alpha} \text{Tr} \ln [\mathbb{A}^\alpha + (\mathbb{I} - \mathbb{A})^\alpha] = \sum_{n=1}^N e_\alpha(a_n), \quad (3.57)$$

¹The case $\mu = 0$ (Neumann boundary conditions at $x = L$) can be treated along the same lines.

a_n being the eigenvalues of the matrix (3.53-3.56). Using (3.57), we will show below that

$$S^{(\alpha)}(T; N) = \mathcal{C}^{(\alpha)}(T) \ln N + O(1), \quad (3.58)$$

with a pre-factor $\mathcal{C}^{(\alpha)}(T)$ that depends on the transmission amplitude and not only on the central charge $c = 1$.

Before considering a generic value of α , it is instructive to discuss the Rényi entropy $\alpha = 2$. In this case Eq. (3.57) takes the simple form

$$S^{(2)}(T; N) = -\text{Tr} \ln [\mathbb{I} - 2\mathbb{E}(T)] = \sum_{k=1}^{\infty} \frac{2^k}{k} \text{Tr} \mathbb{E}^k(T), \quad (3.59)$$

where the combination

$$\mathbb{E}(T) \equiv \mathbb{A}(\mathbb{I} - \mathbb{A}) \quad (3.60)$$

represents the natural variable for performing the computation. The main idea at this point is to reduce the evaluation of (3.59) to the case of full transmission $T = 1$, when the defect is absent and one can use therefore the known [40] behavior of $\text{Tr} \mathbb{E}^k$ for free fermion gas on the interval, namely

$$\text{Tr} \mathbb{E}^k(T = 1) = \frac{1}{2\pi^2} \frac{[(k-1)!]^2}{(2k-1)!} \ln N + O(1). \quad (3.61)$$

For this purpose we first establish the fundamental relation

$$\text{Tr} \mathbb{E}^k(T) = T^{2k} \text{Tr} \mathbb{E}^k(T = 1), \quad (3.62)$$

which captures the impact of the point-like interaction in the junction on the entanglement entropy (3.59). In order to prove (3.62), we exploit the property that a reordering of rows and columns of \mathbb{E} does not change the trace of \mathbb{E}^k , we are interested in. We then write the overlap matrix in the following block form

$$\mathbb{A} = \begin{pmatrix} \frac{\epsilon^2}{1+\epsilon^2} \mathbb{I} & T \mathbb{B}_1 \\ T \mathbb{B}_2 & \frac{1}{1+\epsilon^2} \mathbb{I} \end{pmatrix}, \quad (3.63)$$

with quadratic blocks of size \mathcal{N} . Here \mathbb{I} is the $\mathcal{N} \times \mathcal{N}$ identity matrix and with respect to (3.53-3.56) we have reordered the lines and rows of \mathbb{A} in such a way that the upper block on the left and the lower one on the right coincide with (3.53) and (3.54) respectively (i.e. after this reordering the first lines and rows are the odd indices n, m , while the right-lower block is formed by even n, m). \mathbb{B}_1 and \mathbb{B}_2 are T -independent matrices, whose explicit form is not essential for the proof. Using the representation (3.63) and the relation (3.50), one gets

$$\mathbb{E} = \mathbb{A}(\mathbb{I} - \mathbb{A}) = \begin{pmatrix} \frac{\epsilon^2}{1+\epsilon^2} \mathbb{I} & T \mathbb{B}_1 \\ T \mathbb{B}_2 & \frac{\epsilon^2}{1+\epsilon^2} \mathbb{I} \end{pmatrix} \begin{pmatrix} \frac{1}{1+\epsilon^2} \mathbb{I} & -T \mathbb{B}_1 \\ -T \mathbb{B}_2 & \frac{\epsilon^2}{1+\epsilon^2} \mathbb{I} \end{pmatrix} = \frac{T^2}{4} \begin{pmatrix} \mathbb{I} + \mathbb{B}_1 \mathbb{B}_2 & 0 \\ 0 & \mathbb{I} + \mathbb{B}_2 \mathbb{B}_1 \end{pmatrix}, \quad (3.64)$$

which proves (3.62). Finally, plugging (3.61,3.62) in (3.59) one obtains

$$\mathcal{C}^{(2)}(T) = \sum_{k=1}^{\infty} \frac{2^{k-1}[(k-1)!]^2}{\pi^2 k(2k-1)!} T^{2k} = \frac{2}{\pi^2} \arcsin^2 \left(\frac{T}{\sqrt{2}} \right), \quad (3.65)$$

where we used

$$\sum_{k=1}^{\infty} \frac{[(k-1)!]^2}{(2k)!} (2x)^{2k} = 2 \arcsin^2 x. \quad (3.66)$$

We stress that each integer value of $\alpha \geq 2$ can be treated in analogous way. For $\alpha = 3, 4$ one finds for instance

$$\mathcal{C}^{(3)}(T) = \sum_{k=1}^{\infty} \frac{3^k[(k-1)!]^2}{\pi^2 2k(2k-1)!} T^{2k} = \frac{2}{\pi^2} \arcsin^2 \left(\frac{T\sqrt{3}}{2} \right), \quad (3.67)$$

and

$$\begin{aligned} \mathcal{C}^{(4)}(T) &= \sum_{k=1}^{\infty} \frac{[(2+\sqrt{2})^k + (2-\sqrt{2})^k][(k-1)!]^2}{6\pi^2 k(2k-1)!} T^{2k} = \\ &= \frac{2}{3\pi^2} \left[\arcsin^2 \left(\frac{T}{2} \sqrt{2+\sqrt{2}} \right) + \arcsin^2 \left(\frac{T}{2} \sqrt{2-\sqrt{2}} \right) \right]. \end{aligned} \quad (3.68)$$

3.1.1 Result for generic integer α

In order to obtain the result for generic integer α we first need to write the combination $\mathbb{A}^\alpha + (\mathbb{I} - \mathbb{A})^\alpha$ in terms of the matrix \mathbb{E} . This can be achieved by formally inverting Eq. (3.60) as

$$\mathbb{A} = \frac{1}{2}(1 \pm \sqrt{1 - 4\mathbb{E}}), \quad (3.69)$$

with an ambiguity in the choice of the sign reflecting the degeneration of the spectrum of \mathbb{E} . However, in the needed combination

$$\mathbb{A}^\alpha + (\mathbb{I} - \mathbb{A})^\alpha = 2^{-\alpha} \left[(1 \pm \sqrt{1 - 4\mathbb{E}})^\alpha + (1 \mp \sqrt{1 - 4\mathbb{E}})^\alpha \right], \quad (3.70)$$

the choice of this sign is unimportant. Notice that although the apparent presence of a square-root, for integer α the above expression is a polynomial in \mathbb{E} of degree $\lfloor \alpha/2 \rfloor$, that is the integer part of $\alpha/2$. Using the binomial theorem $(1+x)^\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{k} x^k$, we have

$$\mathbb{A}^\alpha + (\mathbb{I} - \mathbb{A})^\alpha = 2^{-\alpha} \sum_{k=0}^{\alpha} \binom{\alpha}{k} (1 - 4\mathbb{E})^{k/2} (1 - (-1)^k) \quad (3.71)$$

$$= 2^{1-\alpha} \sum_{k=0}^{\lfloor \alpha/2 \rfloor} \binom{\alpha}{2k} (1 - 4\mathbb{E})^k, \quad (3.72)$$

which makes the polynomial form explicit. Using again the binomial theorem for $(1 - 4\mathbb{E})^k$, we have

$$\mathbb{A}^\alpha + (\mathbb{I} - \mathbb{A})^\alpha = 2^{1-\alpha} \sum_{k=0}^{\lfloor \alpha/2 \rfloor} \binom{\alpha}{2k} \sum_{p=0}^k \binom{k}{p} (-4\mathbb{E})^k = - \sum_{p=0}^{\lfloor \alpha/2 \rfloor} v_p (-4\mathbb{E})^p,$$

where we defined

$$v_p \equiv -2^{1-\alpha} \sum_{k=p}^{\lfloor \alpha/2 \rfloor} \binom{\alpha}{2k} \binom{k}{p} = - \binom{\alpha}{2p} \frac{\Gamma(\alpha - p) \Gamma(p + 1/2)}{\sqrt{\pi} \Gamma(\alpha)}. \quad (3.73)$$

In order to calculate the Rényi entropies, we expand in series of \mathbb{E} the quantity $\ln[\mathbb{A}^\alpha + (1 - \mathbb{A})^\alpha]$, obtaining

$$\ln[\mathbb{A}^\alpha + (1 - \mathbb{A})^\alpha] = \ln[1 - \sum_{p=1}^{\lfloor \alpha/2 \rfloor} v_p (-4\mathbb{E})^p] = - \sum_{j=1}^{\infty} \frac{1}{j} \left[\sum_{p=1}^{\lfloor \alpha/2 \rfloor} v_p (-4\mathbb{E})^p \right]^j. \quad (3.74)$$

Using now the multinomial identity we have

$$\ln[\mathbb{A}^\alpha + (1 - \mathbb{A})^\alpha] = - \sum_{j=1}^{\infty} \frac{1}{j} \sum'_{k_i} \frac{j!}{k_1! \dots k_{\lfloor \alpha/2 \rfloor}!} \prod_{p=1}^{\lfloor \alpha/2 \rfloor} [v_p (-4\mathbb{E})^p]^{k_p} \quad (3.75)$$

where we introduced the symbol \sum'_{k_i} for the constrained sum $\sum_{k_1, \dots, k_{\lfloor \alpha/2 \rfloor}}$ with $\sum k_i = j$. We now introduce a sum over K which will be equal to $\sum p k_p$ with the help of a Kronecker delta:

$$\ln[\mathbb{A}^\alpha + (1 - \mathbb{A})^\alpha] = - \sum_{K=1}^{\infty} \sum_{j=1}^{\infty} \sum'_{k_i} \frac{(j-1)!}{k_1! \dots k_{\lfloor \alpha/2 \rfloor}!} (-4\mathbb{E})^K \delta_{K, \sum_p p k_p} \prod_{p=1}^{\lfloor \alpha/2 \rfloor} v_p^{k_p}. \quad (3.76)$$

Using the contour integral representation of the Kronecker delta over the unitary circle $|z| = 1$

$$\delta_{a,b} = \frac{1}{2\pi i} \oint dz z^{a-b-1}, \quad (3.77)$$

we have that the above expression equals

$$\begin{aligned} & \frac{1}{2\pi i} \oint dz \sum_{K=1}^{\infty} (-4\mathbb{E})^K z^{K-1} \sum_{j=1}^{\infty} \frac{1}{j} \sum'_{k_i} \frac{j!}{k_1! \dots k_{\lfloor \alpha/2 \rfloor}!} \prod_{p=1}^{\lfloor \alpha/2 \rfloor} (v_p z^{-p})^{k_p} \\ &= \frac{1}{2\pi i} \sum_{K=1}^{\infty} (-4\mathbb{E})^K \oint dz z^{K-1} \ln(1 + \sum_{p=1}^{\lfloor \alpha/2 \rfloor} v_p z^{-p}) \\ &= \frac{1}{2\pi i} \sum_{K=1}^{\infty} (-4\mathbb{E})^K \oint dz z^{K-1} \ln \left[\frac{(1 + \sqrt{1 + z^{-1}})^\alpha + (1 - \sqrt{1 + z^{-1}})^\alpha}{2^\alpha} \right], \end{aligned} \quad (3.78)$$

where, in the last line, we recognized the expansion of the function from which we started from for a complex argument (no matrices).

The integral over z can be performed with standard techniques of integrals on the complex plane. The various contributions come from the discontinuities at the cuts which are placed between the zeros of the argument of the logarithm, i.e. at the z_p satisfying

$$\left(1 + \sqrt{1 + z_p^{-1}}\right)^\alpha + \left(1 - \sqrt{1 + z_p^{-1}}\right)^\alpha = 0. \quad (3.79)$$

All the solutions of this equation are simply found as

$$z_p = -\cos^2 \frac{\pi(2p-1)}{2\alpha}, \quad \text{with } p = 1, \dots, \alpha/2. \quad (3.80)$$

Thus the integral is given by

$$\frac{1}{2\pi i} \oint dz z^{K-1} \ln \left[\frac{(1 + \sqrt{1 + z^{-1}})^\alpha + (1 - (\sqrt{1 + z^{-1}})^\alpha)}{2^\alpha} \right] = \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \frac{z_p^K}{K}, \quad (3.81)$$

implying

$$\ln[\mathbb{A}^\alpha + (1 - \mathbb{A})^\alpha] = - \sum_{K=1}^{\infty} (-4\mathbb{E})^K \frac{(-1)^K}{K} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \cos^{2K} \left(\pi \frac{2p-1}{2\alpha} \right). \quad (3.82)$$

In this final form, it is straightforward to take the trace using Eq. (3.62) to obtain

$$\text{Tr} \ln[\mathbb{A}^\alpha + (1 - \mathbb{A})^\alpha] = -\frac{1}{2\pi^2} \ln N \sum_{K=1}^{\infty} T^{2K} 4^K \frac{[(K-1)!]^2}{(2K-1)!} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \cos^{2K} \left(\pi \frac{2p-1}{2\alpha} \right). \quad (3.83)$$

Inverting the order of the sums, the sum over K can be now performed using Eq. (3.66) and we have

$$\text{Tr} \ln[\mathbb{A}^\alpha + (1 - \mathbb{A})^\alpha] = -\ln N \frac{2}{\pi^2} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \arcsin^2 \left[T \cos \frac{(2p-1)\pi}{2\alpha} \right], \quad (3.84)$$

that leads to the coefficient

$$\mathcal{C}^{(\alpha)}(T) = \frac{1}{\alpha-1} \frac{2}{\pi^2} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \arcsin^2 \left[T \cos \frac{(2p-1)\pi}{2\alpha} \right]. \quad (3.85)$$

For $\alpha = 2, 3, 4$ it coincides with the result reported in the previous subsection.

We observe that $\mathcal{C}^{(\alpha)}(0) = 0$ and $\mathcal{C}^{(\alpha)}(1)$ provide useful checks. The value $T = 0$ describes a totally reflecting defect; the two edges are isolated and indeed the entanglement vanishes. For $T = 1$ we have

$$\mathcal{C}^{(\alpha)}(T=1) = \frac{1}{\alpha-1} \frac{2}{\pi^2} \sum_{p=1}^{\lfloor \alpha/2 \rfloor} \left[\frac{\pi}{2} - \frac{(2p-1)\pi}{2\alpha} \right]^2 = \frac{1}{12} \left(1 + \frac{1}{\alpha} \right). \quad (3.86)$$

This corresponds to full transmission, i.e. the defect is absent and one is considering the entanglement entropy of half of a system of length $2L$ with boundaries obtained in [38] and compatible with the standard CFT result [27].

3.1.2 The analytic continuation

In order to investigate non-integer values of α , we exploit the results of Ref. [35, 36] for the Ising and XX spin-chains. Using methods based on corner transfer matrix and conformal mappings, Eisler and Peschel derived the entanglement entropy of the subsystem on (let us say) the right of the defect, for a chain of length $2L$ with a defect in the middle. For the XX chain, that after a Jordan-Wigner transformation corresponds to a lattice gas of free spinless fermions, the result can be written in the form [35]

$$S^{(\alpha)}(T; L) = \mathcal{C}_L^{(\alpha)}(T) \ln L + O(1), \quad (3.87)$$

with $\mathcal{C}_L^{(\alpha)}(T)$ given by

$$\mathcal{C}_L^{(\alpha)}(T) = \frac{2}{\pi^2(1-\alpha)} \int_0^\infty dx \ln \left[\frac{1 + e^{-2\alpha\omega(x,T)}}{(1 + e^{-2\omega(x,T)})^\alpha} \right], \quad (3.88)$$

where

$$\omega(x, T) = \text{acosh} \left[\frac{\cosh(x)}{T} \right]. \quad (3.89)$$

Actually only the result for $\alpha = 1$ has been reported in Ref. [35], but the derivation for general α is straightforward from the results reported there. A result of $\alpha = 2$ has been reported in [36] and for general integer α in [46]. For a simple comparison between our work and Ref. [35], we mention that the transmission amplitude T in Ref. [35] is called s .

For integer α , it is straightforward to check numerically that $\mathcal{C}_L^{(\alpha)}(T)$ and $\mathcal{C}^{(\alpha)}(T)$ in Eq. (3.85) are equal, as it is possible to show analytically [46]. This coincidence does not come unexpected. Indeed, since Eq. (3.87) is valid for finite density $N/2L = 1/2$ on the lattice, it can be turned in the entanglement entropy as function of N , simply by replacing L with N , but assuming (as we did in [38] on the basis of numerical data) that the dependence on T is universal. Taking now the continuum limit, we straightforwardly deduce that

$$\mathcal{C}^{(\alpha)}(T) = \mathcal{C}_L^{(\alpha)}(T). \quad (3.90)$$

However, the above computation proves the universal T dependence with no assumption. The computation of Ref. [46] also shows that the result in Eq. (3.88) is the analytic continuation of Eq. (3.85) to non-integer values of α . In particular, in the von Neumann case ($\alpha = 1$), the integral can be performed and one has [35]

$$\mathcal{C}_L^{(1)}(T) = \frac{1}{\pi^2} \{ [(1+T) \ln(1+T) + (1-T) \ln(1-T)] \ln T + (1+T) \text{Li}_2(-T) + (1-T) \text{Li}_2(T) \}. \quad (3.91)$$

3.1.3 Comparison with numerical computation

We now turn to briefly present the numerical data for the entanglement entropies which have been fundamental for the conceptual understanding that led to the exact computation in the

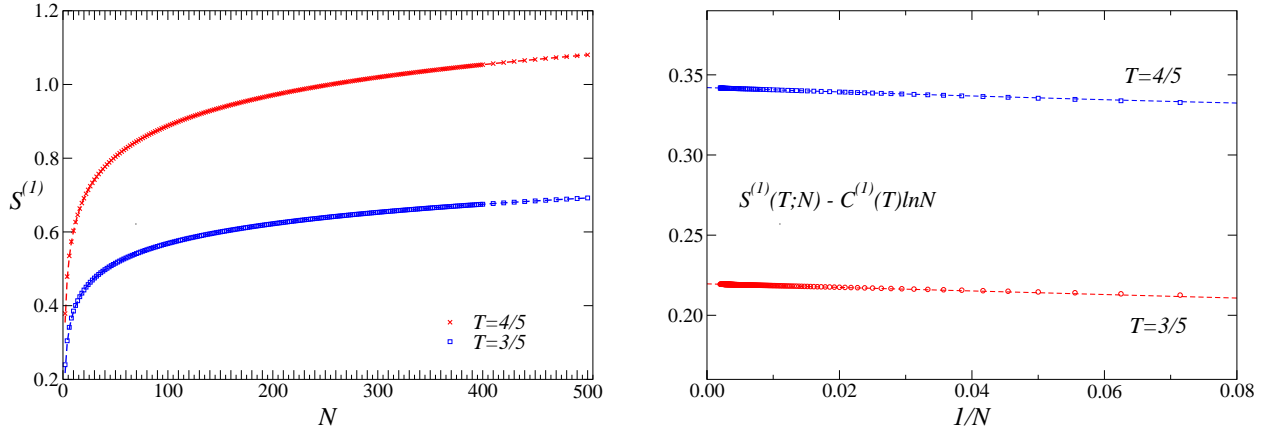


Figure 2: (Color online) The von Neumann edge entanglement entropy in the two-wire junction for $T = 3/5$ and $T = 4/5$ where we considered only even values of N . Left: We plot $S^{(1)}(T, N)$ vs N . Right: The subtracted quantity $S^{(1)}(T, N) - \mathcal{C}^{(1)}(T) \ln N$, where $\mathcal{C}^{(1)}(T)$ is given by Eq. (3.88). The lines show fits of the data for $N \gtrsim 400$, where the correction to the leading behavior $\mathcal{C}^{(1)}(T) \ln N$ is a polynomial $b_0 + b_1/N + b_2/N^2$.

previous subsection. Furthermore numerical computations give non trivial insights about the corrections to the asymptotic behavior. In particular we anticipate that in analogy to what found for the half-space Rényi entanglement entropies in homogeneous systems [39, 47], the leading suppressed corrections turns out to be $O(N^{-1/\alpha})$.

The numerical estimates of the factor $\mathcal{C}^{(1)}(T)$, obtained from our results for the entanglement entropy up to $N \approx 500$, perfectly match the function (3.91), within a precision better than $O(10^{-6})$. Fig. 2 shows the results for $T = 3/5$ and $T = 4/5$ for even N . Fits to the $T = 4/5$ data for $400 \lesssim N \leq 500$

$$S^{(1)}(T; N) = a \ln N + b_0 + b_1/N + b_2/N^2 + b_3/N^3 \quad (3.92)$$

give $a = 0.118841$, $b_0 = 0.342056$, $b_1 = -0.1404$, etc..., where a should be compared with $\mathcal{C}^{(1)}(4/5) = 0.11884065\dots$. Analogous results are obtained for $T = 3/5$; we find $a = 0.076078$ to be compared with $\mathcal{C}^{(1)}(3/5) = 0.07607750$.

A complete agreement is also found for the Rényi entropies. For $\alpha = 2$ for instance, the data fit the asymptotic behavior

$$S^{(2)}(T; N) = \mathcal{C}^{(2)}(T) \ln N + b_0 + b_1/N^{1/2} + b_2/N + b_3/N^{3/2} + \dots \quad (3.93)$$

where the leading coefficient $\mathcal{C}^{(2)}(T)$ is given by Eq. (3.65), as shown by Fig. 3.

In both Eqs. (3.92) and (3.93) we use for the subleading corrections the form $O(N^{-1/\alpha})$. We mention that considering also odd values of the particles number N , we observe also in the graph entanglement entropies corrections to the scaling which depends on the parity of N ,

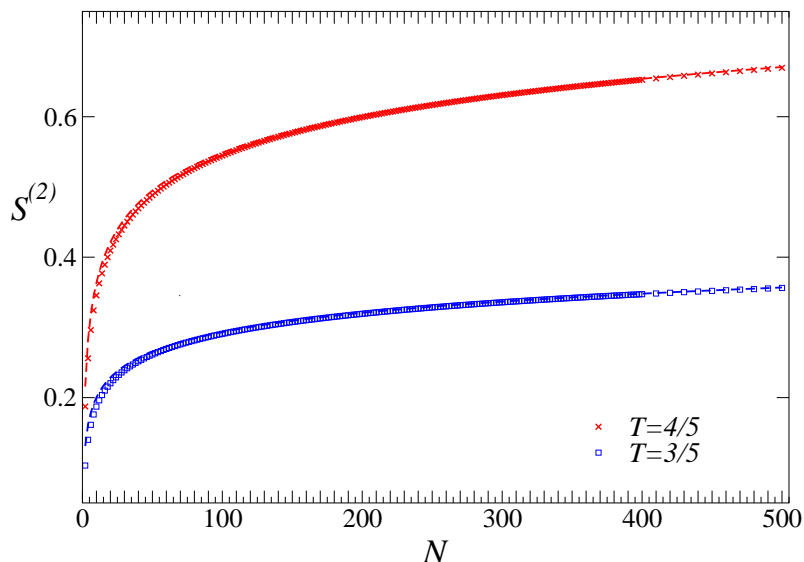


Figure 3: (Color online) The $\alpha = 2$ Rényi edge entanglement entropy $S^{(2)}(T, N)$ in the two-wire junction for $T = 3/5$ and $T = 4/5$ for even number of particles N . The line show the curve $\mathcal{C}^{(2)}(T) \ln N + b_0 + b_1/N^{1/2} + b_2/N$ where the coefficients b_i are fitted using the data for $N \gtrsim 400$.

in analogy to what observed in the absence of defects both in the continuum [39] and on the lattice [47].

3.2 The case of $M > 2$ edges

The generalization to $M > 2$ edges is now straightforward. The novelty is that now the entanglement entropy on the edge i with respect to the whole junction depends on i . As expected from (2.42), this dependence is encoded in the transmission amplitudes T_i , which for $M > 2$ do not coincide in general for different i . In order to investigate this aspect, it is convenient to introduce the variable

$$\Upsilon_i^2 = \frac{1}{2} \left(1 + \sqrt{1 - T_i^2} \right). \quad (3.94)$$

The two point-function (2.42) takes now the form

$$C_{ii}^N(x, y) = \Upsilon_i^2 \sum_{n=1}^{\mathcal{N}} f_+(x, n) f_+(y, n) + (1 - \Upsilon_i^2) \sum_{n=1}^{\mathcal{N}} f_-(x, n) f_-(y, n). \quad (3.95)$$

Equivalently, using (2.38,2.39), one finds in the case $\mu = \infty$

$$C_{ii}^N(x, y) = \sum_{n=1}^{2\mathcal{N}} \bar{\chi}_i(x, n) \chi_i(y, n), \quad (3.96)$$

with

$$\chi_i(k, x) = \begin{cases} \Upsilon_i \sqrt{2} \cos(k\pi \frac{x}{2}), & k = 1, 3, \dots, 2\mathcal{N} - 1, \\ \sqrt{1 - \Upsilon_i^2} \sqrt{2} \sin(k\pi \frac{x}{2}), & k = 2, 4, \dots, 2\mathcal{N}. \end{cases} \quad (3.97)$$

The reduced overlap matrix \mathbb{A} now also carries an edge index $i = 1, \dots, M$ and is given by

$$\mathbb{A}_{mn}^{(i)} = 2\Upsilon_i^2 \delta_{mn}, \quad m, n - \text{odd}, \quad (3.98)$$

$$\mathbb{A}_{mn}^{(i)} = 2(1 - \Upsilon_i^2) \delta_{mn}, \quad m, n - \text{even}, \quad (3.99)$$

$$\mathbb{A}_{mn}^{(i)} = 2\Upsilon_i \sqrt{1 - \Upsilon_i^2} \frac{2n}{\pi(n^2 - m^2)}, \quad m - \text{odd}, n - \text{even}, \quad (3.100)$$

$$\mathbb{A}_{mn}^{(i)} = 2\Upsilon_i \sqrt{1 - \Upsilon_i^2} \frac{2m}{\pi(m^2 - n^2)}, \quad m - \text{even}, n - \text{odd}. \quad (3.101)$$

Comparing the matrices (3.53-3.56) and (3.98-3.101) one concludes that the entanglement entropy $S_i^{(\alpha)}$ of the edge i with respect to the whole junction depends on the edge via the transmission amplitude

$$T_i = 2\Upsilon_i \sqrt{1 - \Upsilon_i^2} \equiv \sqrt{1 - \mathbb{S}_{ii}^2} \quad (3.102)$$

and can be expressed by

$$S_i^{(\alpha)}(T_i; N) = S^{(\alpha)}(T_i; N), \quad (3.103)$$

where $S^{(\alpha)}$ is the entropy (3.57) in the case $M = 2$. Therefore, the asymptotic behavior for large N is given by

$$S_i^{(\alpha)}(T_i; N) = \mathcal{C}^{(\alpha)}(T_i) \ln N + O(1), \quad (3.104)$$

where $\mathcal{C}^{(\alpha)}$ is the universal function defined by (3.85, 3.88, 3.89). At this point all the numerical results in the previous subsection concerning $\mathcal{C}^{(\alpha)}$ apply.

4 Schrödinger junction with harmonic potential

In this section we consider a star graph Γ with M *infinite* edges and a harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$ trapping the gas in the bulk (harmonic trap). The Schrödinger field $\psi_i(t, x)$ thus satisfies

$$\left(i\partial_t + \frac{1}{2m}\partial_x^2 - \frac{1}{2}m\omega^2 x^2 \right) \psi_i(t, x) = 0, \quad x > 0. \quad (4.105)$$

Since $V(x)$ defines a self-adjoint multiplication operator, the vertex boundary conditions controlling the self-adjointness of the total Hamiltonian $-\partial_x^2 + V(x)$ are still given by (2.2). Accordingly, the critical \mathbb{S} matrices are parametrized by (2.24). The field $\varphi_i(t, x)$, defined by

(2.12), satisfies

$$(\partial_x \varphi_i)(t, 0) = 0, \quad 1 \leq i \leq p, \quad (4.106)$$

$$\varphi_i(t, 0) = 0, \quad p < i \leq M, \quad (4.107)$$

for all t . The eigenfunctions of $-\partial_x^2 + V(x)$, obeying these boundary conditions, are

$$\phi_i(n, x) = \begin{cases} f_+(n, x), & 1 \leq i \leq p, \\ f_-(n, x), & p < i \leq M, \end{cases} \quad (4.108)$$

where $f_{\pm}(n, x)$ are expressed in terms of the Hermite polynomials as follows²

$$f_+(n, x) = \frac{1}{\pi^{1/4} \sqrt{2^{2n-1} (2n)!}} H_{2n}(x) e^{-x^2/2}, \quad n = 0, 1, \dots \quad (4.109)$$

$$f_-(n, x) = \frac{1}{\pi^{1/4} \sqrt{2^{2n} (2n+1)!}} H_{2n+1}(x) e^{-x^2/2}, \quad n = 0, 1, \dots \quad (4.110)$$

Inserting (4.109, 4.110) in (3.95), one gets the two-point correlation function

$$C_{ii}^N(x, y) = \sum_{n=0}^{2\mathcal{N}} \bar{\chi}_i(x, n) \chi_i(y, n), \quad (4.111)$$

with

$$\chi_i(k, x) = \begin{cases} \frac{\Upsilon_i}{\pi^{1/4} \sqrt{2^{k-1} k!}} H_k(x) e^{-x^2/2}, & k = 1, 3, \dots, 2\mathcal{N} - 1, \\ \frac{\sqrt{1 - \Upsilon_i^2}}{\pi^{1/4} \sqrt{2^{k-1} k!}} H_k(x) e^{-x^2/2}, & k = 0, 2, \dots, 2\mathcal{N}. \end{cases} \quad (4.112)$$

At this point one can derive the relative \mathbb{A} -matrix and compute the entanglement entropy. The numerical data, displayed in fig. 4, confirm that in the harmonic case $S_i^{(\alpha)}(T_i; N)$ has precisely the behavior described by equation (3.104). An educated guess for the large- N asymptotic behavior of the edge entanglement entropy is that it is the same as that of the hard-wall case, i.e.

$$S^{(\alpha)}(T; N) = \mathcal{C}^{(\alpha)}(T) \ln N + O(1), \quad (4.113)$$

where the functions $\mathcal{C}^{(\alpha)}$ are the same as those in Eq. (3.88). Difference occurs at the level of the $O(1)$ term, as in the case without defect [38, 48, 49].

In order to check it for nontrivial defects, we have computed the entanglement entropy up to $N \approx 50$ for the two-edge problem, and for two values of T , $T = 3/5$ and $T = 4/5$. The results for the von Neumann entropy are shown in Fig. 4. They are perfectly consistent with the conjecture (4.113). For example a fit of the data for $T = 4/5$ and even N to

$$S^{(1)}(T; N) = a \ln N + b_0 + b_1/N + b_2/N^2 + b_3/N^2 \quad (4.114)$$

give $a = 0.118842$, $b_0 = 0.424421$, etc..., where a should be compared with $\mathcal{C}^{(1)}(T = 4/5) = 0.11884065\dots$

²We set for simplicity $\omega = 1$ and $m = 1$.

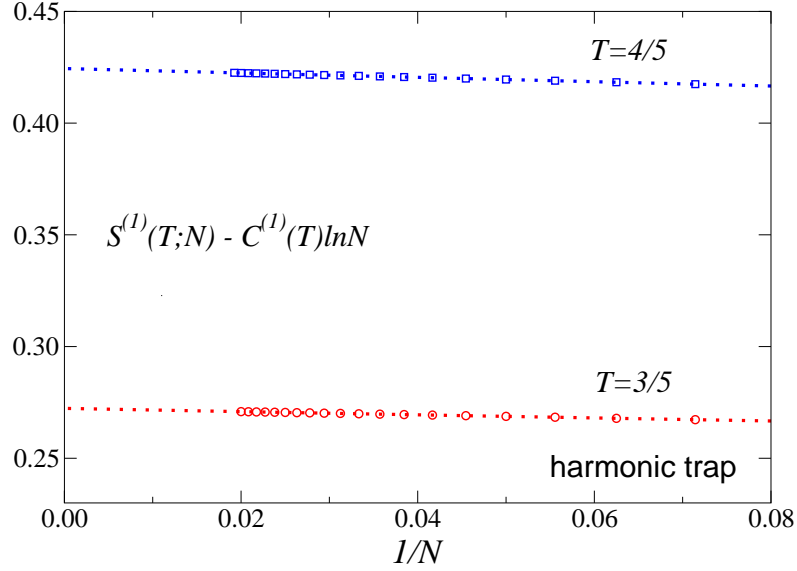


Figure 4: (Color online) The von Neumann edge entanglement entropy in the two-wire junction with an external harmonic potential for $T = 3/5$ and $T = 4/5$. We plot the subtracted quantity $S^{(1)}(T; N) - \mathcal{C}^{(1)}(T) \ln N$, where $\mathcal{C}^{(1)}(T)$ is given by Eq. (3.88). The lines show fits of the data for $N \gtrsim 400$ to a polynomial $b_0 + b_1/N + b_2/N^2$.

5 Outlook and conclusions

We have studied the entanglement entropies of one edge i with respect to the rest of a junction with M edges. Our main result is that when working with a finite number of particles N in edges of finite length L , the Rényi entanglement entropies can be derived analytically, obtaining

$$S^{(\alpha)} = \mathcal{C}^{(\alpha)}(T) \ln N + O(N^0), \quad (5.115)$$

where T is the total transmission probability from the edge i to the rest of the graph and the pre-factor is independent on the number of edges. The analytical computation of $\mathcal{C}^{(\alpha)}$ is given by Eq. (3.85) for integer α and its analytical continuation to non-integer α is given by the prefactor of Eisler and Peschel [35] obtained for the spatial entanglement of a line with a defect reported in Eq. (3.88). Clearly the value of the total transmission does depend on the number of edges and of the kind of junction. The same asymptotic behavior in N also describe the entanglement entropies of systems in which on each arm there is a confining parabolic potential.

We can turn the above asymptotic behavior in a more standard expression for the dependence of the entanglement entropies on the length of the subsystem (i.e. the edge in our case). Indeed, assuming a uniform density of particles we have $N \propto L$ and so

$$S^{(\alpha)} = \mathcal{C}^{(\alpha)}(T) \ln L + O(L^0), \quad (5.116)$$

that is expected to be valid also for lattice models and in particular for M XX spin chains of length L joined at a single common vertex.

Finally, we want to mention that the asymptotic result (5.116) has not yet been derived from conformal field theory. Although it is clear that such derivation must be possible because the CFT encodes all the required ingredients, the practical calculation is very cumbersome, as the similar (but different) result of Ref. [33] shows.

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